COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **23** (4) (2003) 681–691

ON THE FUNCTION $w(x)=|\{1\le s\le k:\ x\equiv a_s\ (\mathrm{mod}\, n_s)\}|$

ZHI-WEI SUN

Received December 21, 1999
Revised November 10, 2000

For a finite system $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ of arithmetic sequences the covering function is $w(x)=|\{1\leqslant s\leqslant k:\ x\equiv a_s\ (\mathrm{mod}\, n_s)\}|$. Using equalities involving roots of unity we characterize those systems with a fixed covering function $w(x)$. From the characterization we reveal some connections between a period n_0 of $w(x)$ and the moduli $n_1,\dots,n_k$ in such a system A. Here are three central results: (a) For each $r=0,1,\dots,n_k/(n_0,n_k)-1$ there exists a $J\subseteq\{1,\dots,k-1\}$ such that $\sum_{s\in J}1/n_s=r/n_k$. (b) If $n_1\leqslant\dots\leqslant n_{k-l}<n_{k-l+1}=\dots=n_k$ $(0<l<k)$, then for any positive integer $r<n_k/n_{k-l}$ with $r\not\equiv 0\ (\mathrm{mod}\, n_k/(n_0,n_k))$, the binomial coefficient $\binom{l}{r}$ can be written as the sum of some (not necessarily distinct) prime divisors of n_k. (c) $\max_{(x\in\mathbb{Z})}w(x)$ can be written in the form $\sum_{(s=1)}^k m_s/n_s$ where $m_1,\dots,m_k$ are positive integers.

1. Introduction

Let $\mathbb{N}=\{0,1,2,\dots\}$ and $\mathbb{Z}^+=\mathbb{N}\setminus\{0\}=\{1,2,3,\dots\}$. For $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$ we put

$$a(n)=a+n\mathbb{Z}=\{\dots,a-2n,a-n,a,a+n,a+2n,\dots\}$$

and call it an arithmetic sequence with modulus n. For a finite system

$$A=\{a_s(n_s)\}_{s=1}^k \tag{1}$$

Mathematics Subject Classification (2000): 11B25; 05A15, 11A07, 11A25, 11B75, 11D68

The research is supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, and the National Natural Science Foundation of P. R. China.

of such sequences we define its *covering function* $w_A:\mathbb{Z}\to\mathbb{N}$ by

$$w_A(x) = |\{1 \leqslant s \leqslant k:\ x \in a_s(n_s)\}|. \tag{2}$$

(Cf. [7].) The function $w_A(x)$ is periodic modulo the least common multiple $[n_1,\dots,n_k]$ of $n_1,\dots,n_k$, also the smallest positive period of $w_A(x)$ divides any period (including $[n_1,\dots,n_k]$) of $w_A(x)$. For a positive integer m, if $w_A(x) \geqslant m$ (resp. $w_A(x) = m$) for all $x \in \mathbb{Z}$ then we say that (1) forms an *m-cover* (resp. *exact m-cover*) (of $\mathbb{Z}$). It is known that $\sum_{s=1}^k 1/n_s \geqslant m$ if (1) is an m-cover, and $\sum_{s=1}^k 1/n_s = m$ if (1) is an exact m-cover. (See, e.g. [13].) Usually one uses *cover* instead of 1-cover, and *disjoint covering system* instead of exact 1-cover. By [18], for each $m = 2,3,4,\dots$ there are exact m-covers no subcover of which is an exact n-cover with $0<n<m$. When the function $w_A(x)$ has a period n_0 relatively prime to some modulus n_t $(1\leqslant t\leqslant k)$, for each $x\in\mathbb{Z}$ there is a $q\in\mathbb{Z}$ such that $a_t+n_tq\equiv x \pmod{n_0}$ and hence $w_A(x)=w_A(a_t+n_tq)>0$, thus (1) forms a cover of $\mathbb{Z}$.

Since P. Erdős ([3]) introduced the notion of cover in 1930's, covers and disjoint covering systems have been widely investigated by various mathematicians. (Cf. R. K. Guy [5] and Š. Porubský and J. Schönheim [9].) Two recent applications can be found in [4] and [17]. The most difficult problem in this area is to describe those moduli $n_1,\dots,n_k$ in a general m-cover (or exact m-cover) (1).

Now we introduce some notations. For $n\in\mathbb{Z}^+$ we set

$$R(n) = \{x \in \mathbb{Z}:\ 0 \leqslant x < n\}, \tag{3}$$

and put

$$D(n) = \Big\{\sum_{p\mid n} px_p:\ x_p \in \mathbb{N} \text{ for any prime divisor } p \text{ of } n\Big\}, \tag{4}$$

in other words $D(n)$ is the additive monoid generated by all the prime factors of n. For $a,b\in\mathbb{Z}$ by (a,b) we mean the greatest common divisor of a and b. For a complex number z, as usual we let

$$\binom{z}{n} = \frac{1}{n!}\prod_{0\leqslant j<n}(z-j) \quad \text{for } n = 0,1,2,\dots\ .$$

(An empty product is taken to be the multiplicative identity 1.) For any α in the field $\mathbb{R}$ of real numbers, we use $[\alpha]$ to denote the greatest integer not exceeding α, also we let $\alpha\mathbb{Z}=\{\alpha x:\ x\in\mathbb{Z}\}$ and $e(\alpha)=e^{2\pi i\alpha}$.

In this paper we fix system (1) and set $[1,k]=\{1,\dots,k\}$. For $I\subseteq[1,k]$ we put $\bar I=[1,k]\setminus I$, and let $[n_s]_{s\in I}$ denote the least common multiple of those n_s with $s\in I$, which is regarded as 1 if $I=\emptyset$.

Let

$$S(A)=\left\{\sum_{s\in J}\frac{1}{n_s}:\ J\subseteq[1,k]\right\}. \tag{5}$$

If system A forms an exact m-cover or an m-cover, what can we say about the set $S(A)$? This question is very challenging and somewhat mysterious, though the author has made a series of investigations (see [12], [13], [14], [15] and [16]). In this paper we aim to tell something about $S(A)$ and the moduli $n_1,\dots,n_k$ according to a period of the function $w_A(x)$.

Now we introduce our main theorems whose proofs will be given later.

Theorem 1.1. *Let $n_0\in\mathbb{Z}^+$ be a period of $w_A(x)$.*

(i) *For $I\subseteq\{1\leqslant s\leqslant k\colon n_s=n\}$ where $n\in\mathbb{Z}^+$, we have*

$$\begin{aligned}(6)\qquad &\left\{\sum_{s\in J}\frac{1}{n_s}:\ J\subseteq\bar I\right\}\\ &\qquad\supseteq\left\{\frac{r}{n}:\ r\in R\left(\frac{n}{(n_0,n)}\right)\ \&\ \binom{|I|+r-1}{r}\notin D(n)\right\}.\end{aligned}$$

(ii) *If $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$, then for each positive integer $r<n_k/n_{k-l}$ either $r\equiv 0 \pmod{n_k/(n_0,n_k)}$ or $\binom{l}{r}\in D(n_k)$.*

In the case $|I|=1$, Theorem 1.1 (i) yields result (a) described in the abstract.

Remark 1.1. Let (1) be an exact m-cover with $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$. In the case $m=1$, a result due to H. Davenport, L. Mirsky, D. Newman and R. Radó asserts that $l>1$, in 1971 M. Newman [6] confirmed a conjecture of Š. Znám by proving that $l\geqslant p$ where p is the smallest prime divisor of n_k. These can also be extended to exact m-covers. (See Porubský [8].) In 1991 the author [11] obtained a general result which implies that $l\geqslant\min_{1\leqslant s\leqslant k-l}n_k/(n_s,n_k)$, in 1995 Y. G. Chen and Š. Porubský [2] showed that

$$l=\sum_{s=1}^{k-l}\frac{n_k}{(n_k,n_s)}x_s\qquad\text{for some } x_1,\dots,x_{k-l}\in\mathbb{N}.$$

(By the way, it should be mentioned that the key equality (3) in [2] first appeared in [11].) It follows that $l\in D(n_k)$. By Theorem 1.1 (ii), actually $\{\binom{l}{r}\colon r\in\mathbb{Z}^+\ \&\ r<n_k/n_{k-l}\}\subseteq D(n_k)$.

Theorem 1.2. *Let I be a subset of $[1,k]$ with $(n_s,n_t)\,|\,a_s-a_t$ for all $s,t\in I$. Then*

$$\left\{\frac{a}{(n_0,[n_s]_{s\in I})}+\sum_{s\in J}\frac{1}{n_s}:\ a\in\mathbb{Z}\ \&\ J\subseteq\bar{I}\right\}\supseteq\frac{1}{[n_s]_{s\in I}}\mathbb{Z} \tag{7}$$

where $n_0\in\mathbb{Z}^+$ is an arbitrary period of $w_A(x)$.

Remark 1.2. Theorem 1.2 in the case $w_A(x)\equiv m$ was obtained by the author [15] in 1997.

Theorem 1.3. *Let $w:\mathbb{Z}\to\mathbb{N}$ be a function with period $n_0\in\mathbb{Z}^+$, and $\{a_s(n_s)\}_{s\in I}$ a finite nonempty system of arithmetic sequences with $|\{s\in I: x\in a_s(n_s)\}|\leqslant w(x)$ for all $x\in\mathbb{Z}$. Then there are $a_0\in R(n_0)$ and $m_s\in\mathbb{Z}^+$ for $s\in I$ such that*

$$a_0+n_0\sum_{s\in I}\frac{m_s}{n_s}=w(1)+\cdots+w(n_0), \tag{8}$$

thus

$$\sum_{s\in I}\frac{m_s}{n_s}=\left\lceil\frac{w(1)+\cdots+w(n_0)}{n_0}\right\rceil \tag{9}$$

providing all the n_s with $s\in I$ are prime to n_0.

Remark 1.3. By Theorem 1.3, $M(A)=\max_{x\in\mathbb{Z}}w_A(x)$ can be written in the form $\sum_{s=1}^k m_s/n_s$ with $m_1,\ldots,m_k\in\mathbb{Z}^+$.

In the next section we will characterize those systems with a fixed covering function, in Section 3 we mainly show Theorems 1.1–1.3.

Let us end this section with the following conjecture.

Conjecture. *Suppose that (1) forms a cover but none of its proper subsystems does. Then we have $|S(A)|\leqslant n_1+\cdots+n_k$, also $S(A)\supseteq\{r/d:\ r\in R(d)\}$ whenever $1/d\in S(A)$.*

2. Systems with a given covering function

In this section we employ roots of unity for our purposes.

Let $I\subseteq[1,k]$. For any c in the rational field $\mathbb{Q}$, we set

$$I^*[c]=\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I}x_s/n_s=c}}e\left(\sum_{s\in I}\frac{a_sx_s}{n_s}\right) \tag{10}$$

and

$$\bar I_*[c] = \sum_{\substack{J\subseteq \bar I\\ \sum_{s\in J}1/n_s=c}} (-1)^{|J|} e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big). \tag{11}$$

(For convenience we think that the equation $\sum_{s\in\emptyset} x_s/n_s=0$ over $\mathbb{N}$ only has the zero solution and so $\emptyset^*[0]=\emptyset_*[0]=1$.)

Lemma 2.1. *Let $w:\mathbb{Z}\to\mathbb{N}$ be a function with period $n_0\in\mathbb{Z}^+$. Put $N=[n_0,n_1,\dots,n_k]$. Then (1) has covering function w if and only if we have the identity*

$$\prod_{s=1}^{k}\Big(1-z^{N/n_s}e^{2\pi i a_s/n_s}\Big)=\prod_{t=1}^{n_0}\Big(1-z^{N/n_0}e^{2\pi i t/n_0}\Big)^{w(t)}. \tag{12}$$

Proof. Notice that all zeroes of either side are Nth roots of unity. For each $x\in\mathbb{Z}$, $e^{-2\pi i x/N}$ is a zero of multiplicity $w_A(x)$ of the left hand side, and a zero of multiplicity $w(x)$ of the right hand side. So, by Viète's theorem, identity (12) holds if and only if $w_A=w$. ∎

Theorem 2.1. *Let $w:\mathbb{Z}\to\mathbb{N}$ be a function having period $n_0\in\mathbb{Z}^+$. Let $I\subseteq[1,k]$ and $v\in\mathbb{Z}$. Then w is the covering function of (1) if and only if*

$$\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar I_*[c-n]=\sum_{a\geqslant 0}(-1)^a I_-[a]I^*\Big[c-\frac{a}{n_0}\Big] \tag{13}$$

holds for all $c\in\mathbb{Q}$, where

$$I_-[a]=\sum_{\substack{v_1,\dots,v_{n_0}\in\mathbb{N}\\ v_1+\cdots+v_{n_0}=a}}\prod_{t=1}^{n_0}\binom{v+w(t)-|I|}{v_t}e\Big(\frac{\sum_{0<t<n_0}tv_t}{n_0}\Big). \tag{14}$$

Remark 2.1. Actually only finite sums are involved in (13).

Proof. Let $N=[n_0,n_1,\dots,n_k]$. For $r\in\mathbb{R}$ with $|r|<1$, let

$$f_1(r)=(1-r^N)^v\prod_{s\in\bar I}\Big(1-r^{N/n_s}e^{2\pi i a_s/n_s}\Big)$$

and

$$f_2(r)=(1-r^N)^{v-|I|}\prod_{s\in I}\frac{1-r^N}{1-r^{N/n_s}e^{2\pi i a_s/n_s}}\times\prod_{t=1}^{n_0}\Big(1-r^{N/n_0}e^{2\pi i t/n_0}\Big)^{w(t)}.$$

By Lemma 2.1, w is the covering function of (1) if and only if for all $-1<r<1$ we have

$$\prod_{s=1}^{k}\left(1-r^{N/n_s}e^{2\pi i a_s/n_s}\right)=\prod_{t=1}^{n_0}\left(1-r^{N/n_0}e^{2\pi it/n_0}\right)^{w(t)},\ \text{i.e. } f_1(r)=f_2(r).$$

Observe that

$$\begin{aligned}
f_1(r)&=\sum_{n=0}^{\infty}(-1)^n\binom{v}{n}r^{nN}\sum_{J\subseteq\bar I}(-1)^{|J|}e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big)r^{\sum_{s\in J}N/n_s}\\
&=\sum_{h=0}^{\infty}r^h\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\sum_{\substack{J\subseteq\bar I\\ n+\sum_{s\in J}1/n_s=h/N}}(-1)^{|J|}e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big)\\
&=\sum_{h=0}^{\infty}r^h\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar I_*\left[\frac{h}{N}-n\right].
\end{aligned}$$

Since $1-r^N=\prod_{t=1}^{n_0}(1-r^{N/n_0}e^{2\pi it/n_0})$, we also have

$$\begin{aligned}
f_2(r)&=\prod_{s\in I}\Big(\sum_{x_s=0}^{n_s-1}r^{Nx_s/n_s}e^{2\pi i a_s x_s/n_s}\Big)\times\prod_{t=1}^{n_0}\left(1-r^{N/n_0}e^{2\pi it/n_0}\right)^{w(t)+v-|I|}\\
&=\sum_{x_s\in R(n_s)\ \text{for}\ s\in I}e\Big(\sum_{s\in I}\frac{a_sx_s}{n_s}\Big)r^{N\sum_{s\in I}x_s/n_s}\\
&\quad\times\prod_{t=1}^{n_0}\sum_{v_t\geqslant 0}(-1)^{v_t}\binom{w(t)+v-|I|}{v_t}e\Big(\frac{tv_t}{n_0}\Big)r^{Nv_t/n_0},
\end{aligned}$$

that is,

$$\begin{aligned}
f_2(r)&=\sum_{h=0}^{\infty}r^h\sum_{a\geqslant 0}\sum_{\substack{x_s\in R(n_s)\ \text{for}\ s\in I\\ \sum_{s\in I}x_s/n_s=h/N-a/n_0}}e\Big(\sum_{s\in I}\frac{a_sx_s}{n_s}\Big)\\
&\quad\times\sum_{\substack{v_1,\ldots,v_{n_0}\in\mathbb{N}\\ v_1+\cdots+v_{n_0}=a}}(-1)^{\sum_{t=1}^{n_0}v_t}\prod_{t=1}^{n_0}\binom{v+w(t)-|I|}{v_t}e\Big(\frac{\sum_{t=1}^{n_0}tv_t}{n_0}\Big)\\
&=\sum_{h=0}^{\infty}r^h\sum_{a\geqslant 0}(-1)^aI_-[a]I^*\left[\frac{h}{N}-\frac{a}{n_0}\right].
\end{aligned}$$

Thus, for (1) to have covering function w, it is necessary and sufficient that

$$\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar{I}_*\left[\frac{h}{N}-n\right]=\sum_{a\geqslant 0}(-1)^a I_-[a]I^*\left[\frac{h}{N}-\frac{a}{n_0}\right]$$

for every $h\in\mathbb{N}$. Clearly both sides of (13) vanish if $cN\notin\mathbb{N}$, so the desired result follows. ∎

Corollary 2.1. *Let $n_0\in\mathbb{Z}^+$ be a period of $w_A(x)$. Let I be a subset of $[1,k]$ and c be a rational with denominator d. Suppose that $I^*[c-n/d_0]=0$ for every positive integer n where $d_0=(n_0,[d,[n_s]_{s\in I}])$. Then $\bar{I}_*[c]=I^*[c]$.*

Proof. Assume that a is a positive integer with $I^*[c-a/n_0]\neq 0$. Then $\sum_{s\in I}x_s/n_s=c-a/n_0$ for some $x_s\in R(n_s)$ $(s\in I)$, and hence $\frac{a}{n_0}[d,[n_s]_{s\in I}]\in\mathbb{Z}$. Since d_0 can be written as $n_0x+[d,[n_s]_{s\in I}]y$ with $x,y\in\mathbb{Z}$, we have $a/n_0=n/d_0$ for some $n\in\mathbb{Z}^+$. This contradicts our supposition.

In view of the above and Theorem 2.1,

$$\bar{I}_*[c]=\sum_{n\geqslant 0}(-1)^n\binom{0}{n}\bar{I}_*[c-n]=(-1)^0 I^*\left[c-\frac{0}{n_0}\right]=I^*[c].$$

This ends the proof. ∎

Corollary 2.2. *Let $I\subseteq[1,k]$, $m\in\mathbb{Z}^+$ and $v\in\mathbb{Z}$. Then (1) forms an exact m-cover if and only if for all $c\in\mathbb{Q}$ we have*

$$\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar{I}_*[c-n]=\sum_{n\geqslant 0}(-1)^n\binom{v+m-|I|}{n}I^*[c-n]. \tag{15}$$

Proof. The constant function $w:\mathbb{Z}\to\mathbb{N}$ given by $w(x)\equiv m$ has period $n_0=1$. For $a\in\mathbb{N}$ clearly $I_-[a]=\binom{v+m-|I|}{a}$. So the desired result follows from Theorem 2.1. ∎

Remark 2.2. In the cases $v=0$ and $v=|I|-m$, (15) turns out to be

$$\sum_{n\geqslant 0}(-1)^n\binom{m-|I|}{n}I^*[c-n]=\bar{I}_*[c] \tag{16}$$

and

$$\sum_{n\geqslant 0}(-1)^n\binom{|I|-m}{n}\bar{I}_*[c-n]=I^*[c] \tag{17}$$

respectively.

3. Proofs of Theorems 1.1–1.3

For $I\subseteq[1,k]$ and $c\in\mathbb{Q}$ we set

$$I^*(c)=\left|\left\{\text{vector } \langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\text{ for } s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}=c\right\}\right| \tag{18}$$

and

$$\bar{I}_*(c)=\left|\left\{J\subseteq\bar{I}:\ \sum_{s\in J}\frac{1}{n_s}=c\right\}\right|=\left|\left\{\langle\delta_s\rangle_{s\in\bar{I}}:\ \delta_s\in\{0,1\}\ \&\ \sum_{s\in\bar{I}}\frac{\delta_s}{n_s}=c\right\}\right|. \tag{19}$$

In this section we will get rid of roots of unity to obtain relations between $I^*(c)$ and $\bar{I}_*(c)$, and then give proofs of our theorems stated in the first section.

Lemma 3.1. *Let $\lambda_1,\ldots,\lambda_k$ be nth roots of unity and $c_1,\ldots,c_k$ nonnegative integers such that $c_1\lambda_1+\cdots+c_k\lambda_k=0$. Then $c_1+\cdots+c_k\in D(n)$.*

Proof. Let a be any integer divisible by none of the prime divisors of n. Then a is prime to n. Applying the automorphism σ_a of the cyclotomic field $\mathbb{Q}(e^{2\pi i/n})$ with $\sigma_a(e^{2\pi i/n})=e^{2\pi ia/n}$ we obtain that $c_1\lambda_1^a+\cdots+c_k\lambda_k^a=0$. Thus $c_1+\cdots+c_k\in D(n)$ by Lemma 9 of [14] (thanks to Chen's ingenious idea in [2]). ∎

Theorem 3.1. *Let $n_0\in\mathbb{Z}^+$ be a period of $w_A(x)$ and I be a subset of $[1,k]$. Let $r\in\mathbb{Z}$ and*

$$c=\min\left\{\sum_{s\in I}\frac{x_s}{n_s}:\ x_s\in\mathbb{N}\text{ for } s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}-\frac{r}{[n_s]_{s\in I}}\in\frac{1}{(n_0,[n_s]_{s\in I})}\mathbb{Z}\right\}. \tag{20}$$

(i) *If $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$, then $\bar{I}_*(c)\geqslant I^*(c)>0$.*
(ii) *If $\bar{I}_*(c)=0$ then $I^*(c)\in D([n_s]_{s\in I})$.*

Proof. The existence of c follows from Proposition 2.2 of [15]. Note that $I^*(c)>0$. For $n\in\mathbb{Z}^+$ clearly $I^*(c-n/(n_0,[n_s]_{s\in I}))=0$ and hence $I^*[c-n/(n_0,[n_s]_{s\in I})]=0$. By Corollary 2.1, $\bar{I}_*[c]=I^*[c]$.

i) When $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$, by the Chinese Remainder Theorem in general form (see Lemma 1 of [10]) there exists an integer a_I congruent to a_s mod n_s for all $s\in I$, therefore

$$I^*[c]=\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I}x_s/n_s=c}} e\left(\sum_{s\in I}a_I\frac{x_s}{n_s}\right)=e(a_Ic)I^*(c)$$

and hence $\bar{I}_*(c)\geqslant|\bar{I}_*[c]|=|I^*[c]|=I^*(c)>0$.

ii) As $\bar{I}_*(c)=0$ we have $I^*[c]=\bar{I}_*[c]=0$, so $I^*(c)\in D([n_s]_{s\in I})$ by Lemma 3.1.

The proof is now complete. ∎

Proof of Theorem 1.1. i) As $0\in D(n)$ the case $I=0$ is trivial. Let $I\neq 0$ and $r\in R(n/(n_0,n))$. It is well known in combinatorics that the equation $\sum_{s\in I}x_s=r$ has exactly $\binom{|I|+r-1}{r}$ solutions over $\mathbb{N}$. (Cf. p.38 of [1].) By Theorem 3.1 (ii), either $I^*(r/n)=\binom{|I|+r-1}{r}\in D(n)$ or $\sum_{s\in J}1/n_s=r/n$ for some $J\subseteq\bar{I}$. This proves part (i).

ii) Let r be an integer with $1\leqslant r<n_k/n_{k-l}$ and $r\not\equiv 0 \ (\mathrm{mod}\ n_k/(n_0,n_k))$. If $1\leqslant j\leqslant k-l$ then $1/n_j\geqslant 1/n_{k-l}>r/n_k$. So, for any $J\subseteq[1,k]$ with $\sum_{s\in J}1/n_s=r/n_k$, we have $J\subseteq\{k-l+1,\dots,k\}$ and $|J|=r$. Since $\emptyset^*[r/n_k-n/(n_0,n_k)]=0$ for all $n\in\mathbb{N}$, applying Corollary 2.1 we find that

$$\sum_{\substack{J\subseteq\{k-l+1,\dots,k\}\\ |J|=r}} e\left(\frac{\sum_{s\in J}a_s}{n_k}\right)=\sum_{\substack{J\subseteq[1,k]\\ \sum_{s\in J}1/n_s=r/n_k}}(-1)^{|J|-r}e\left(\sum_{s\in J}\frac{a_s}{n_s}\right)=0.$$

Thus, with the help of Lemma 3.1,

$$\binom{l}{r}=|\{J\subseteq\{k-l+1,\dots,k\}:\ |J|=r\}|\in D(n_k).$$

This concludes the proof. ∎

Proof of Theorem 1.2. Let r be any integer and c be as in (20). Then $c+a/(n_0,[n_s]_{s\in I})=r/[n_s]_{s\in I}$ for some $a\in\mathbb{Z}$. By Theorem 3.1 (i), there is a $J\subseteq\bar{I}$ such that $\sum_{s\in J}1/n_s=c$. So we have the desired result. ∎

Proof of Theorem 1.3. By adding some residue classes modulo $N=[n_0,[n_s]_{s\in I}]$ we can extend $\{a_s(n_s)\}_{s\in I}$ to a system having covering function $w(x)$. Without any loss of generality, we may suppose that $I\subseteq[1,k]$ and (1) has covering function $w(x)$. Let $c=\sum_{s\in\bar{I}}1/n_s$. Then $\bar{I}_*[c]\neq 0$. Applying Theorem 2.1 with $v=0$ we find that $I^*[c-a/n_0]\neq 0$ for some $a\in\mathbb{N}$. So there are $x_s\in R(n_s)$ for $s\in I$ such that $c-a/n_0=\sum_{s\in I}x_s/n_s$. Note that

$$c+\sum_{s\in I}\frac{1}{n_s}=\sum_{s=1}^{k}\frac{1}{n_s}=\frac{1}{n_0}\sum_{t=1}^{n_0}w(t)$$

where in the last step we calculate the degrees of both sides of (12). Write $a = a_0 + n_0 q$ where $a_0 \in R(n_0)$ and $q \in \mathbb{N}$. Then

$$\frac{1}{n_0}\sum_{t=1}^{n_0} w(t) - \frac{a_0}{n_0} = q + \sum_{s\in I}\frac{x_s+1}{n_s} = \sum_{s\in I}\frac{m_s}{n_s}$$

where those m_s with $s \in I$ are suitable positive integers. If all the n_s with $s \in I$ are prime to n_0, then n_0 must divide $\sum_{t=1}^{n_0} w(t) - a_0$ and hence

$$\sum_{s\in I}\frac{m_s}{n_s} = \frac{w(1)+\cdots+w(n_0)-a_0}{n_0} = \left[\frac{w(1)+\cdots+w(n_0)}{n_0}\right].$$

We are done. ∎

Acknowledgements. The author is very grateful to the three referees for their valuable suggestions on the first version of the paper.

References

[1] R. A. Brualdi: *Introductory Combinatorics*, Elsevier, Amsterdam, 1977.

[2] Y. G. Chen and Š. Porubský: Remarks on systems of congruence classes, *Acta Arith.* **71** (1995), 1–10. MR 96j:11014.

[3] P. Erdős: On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.* **2** (1950), 113–123. MR 13, 437.

[4] A. Granville and K. Soundararajan: A binary additive problem of Erdős and the order of 2 mod p^2, *Ramanujan J.* **2** (1998), 283–298.

[5] R. K. Guy: *Unsolved Problems in Number Theory* (2nd edition), Springer-Verlag, New York, 1994, Sections F13 and F14. MR 96e:11002. [The third version is now in press.]

[6] M. Newman: Roots of unity and covering sets, *Math. Ann.* **191** (1971), 279–282. MR 44#3972.

[7] Š. Porubský: Covering systems and generating functions, *Acta Arith.* **26** (1974), 223–231. MR 51#328.

[8] Š. Porubský: On m times covering systems of congruences, *Acta Arith.* **29** (1976), 159–169. MR 53#2884.

[9] Š. Porubský and J. Schönheim: Covering system of Paul Erdős: past, present and future, *in: Paul Erdős and his Mathematics* (edited by G. Halász, L. Lovász, M. Simonovits, V. T. Sós), Bolyai Soc. Math. Studies 11., Budapest, 2002, Vol. I, pp. 581–627.

[10] Z. W. Sun: A necessary and sufficient condition for two linear diophantine equations to have a common solution, *Nanjing Univ. J. Natur. Sci.* **25** (1989), no. 1, 10–17. MR 90i:11026.

[11] Z. W. Sun: An imporement to the Znám–Newman result, *Chinese Quart. J. Math.* **6** (1991), no. 3, 90–96.

[12] Z. W. Sun: On exactly m times covers, *Israel J. Math.* **77** (1992), 345–348. MR 93k:11007.

[13] Z. W. Sun: Covering the integers by arithmetic sequences, *Acta Arith.* **72** (1995), 109–129. MR 96k:11013.
[14] Z. W. Sun: Covering the integers by arithmetic sequences II, *Trans. Amer. Math. Soc.* **348** (1996), 4279–4320. MR 97c:11011.
[15] Z. W. Sun: Exact m-covers and the linear form $\sum_{s=1}^{k} x_s/n_s$, *Acta Arith.* **81** (1997), 175–198. MR 98h:11019.
[16] Z. W. Sun: On covering multiplicity, *Proc. Amer. Math. Soc.* **127** (1999), 1293–1300. MR 99h:11012.
[17] Z. W. Sun: Unification of zero-sum problems, subset sums and covers of $\mathbb{Z}$, *Electron. Res. Announc. Amer. Math. Soc.* **9** (2003), 51–60.
[18] M. Z. Zhang: On irreducible exactly m times covering system of residue classes, *J. Sichuan Univ. (Nat. Sci. Ed.)* **28** (1991), 403–408. MR 92j:11001.

Zhi-Wei Sun

Department of Mathematics
Nanjing University
Nanjing 210093
The People's Republic of China
zwsun@nju.edu.cn
http://pweb.nju.edu.cn/zwsun